

# Fourier Analysis

Feb 15, 2022

## Review:

Let  $P_r(x)$  be the Poisson kernel on the circle, i.e.

$$P_r(x) = \frac{1-r^2}{1-2r \cos x + r^2}, \quad 0 \leq r < 1, \quad x \in [-\pi, \pi].$$

Let  $f$  be integrable on the circle, and set

$$u(r, \theta) := P_r * f(\theta), \quad 0 \leq r < 1, \quad \theta \in [-\pi, \pi].$$

Then

- $\Delta u = 0$  on  $\{(r, \theta) : 0 \leq r < 1\}$

- $u(r, \theta) \xrightarrow{r \rightarrow 1} f(\theta)$  if  $f$  is cts at  $\theta$

## Chap 3. Convergence of Fourier series.

### 1. Mean convergence of Fourier series

We are going to prove the following results.

Thm 1.1: Let  $f$  be integrable on the circle. Then

$$(1) \int_{-\pi}^{\pi} |f(x) - S_N f(x)|^2 dx \rightarrow 0 \text{ as } N \rightarrow \infty$$

( $L^2$ -convergence)

(2) (Parseval identity)

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$$

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Let  $\mathcal{R}$  denote the collection of all  $\mathbb{C}$ -valued integrable functions on the circle. Then  $\mathcal{R}$  is a linear space over  $\mathbb{C}$ .

- If  $f \in \mathcal{R}$  and  $c \in \mathbb{C}$ , then  $c \cdot f \in \mathcal{R}$ .
- If  $f, g \in \mathcal{R}$ , then  $f + g \in \mathcal{R}$ .

Let  $f, g \in \mathcal{R}$ . Define

$$\langle f, g \rangle := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} dx$$

Indeed,  $\langle \cdot, \cdot \rangle$  is an inner product over  $\mathbb{C}$ , which satisfies the following 3 properties.

① (Conjugate symmetry)

$$\langle f, g \rangle = \overline{\langle g, f \rangle}$$

② (Linearity in the first argument)

$$\langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$$

$$\alpha, \beta \in \mathbb{C}, f, g, h \in \mathcal{R}.$$

(conjugate linearity in the second argument)

$$\langle f, \alpha g + \beta h \rangle = \bar{\alpha} \langle f, g \rangle + \bar{\beta} \langle f, h \rangle$$

③  $\langle f, f \rangle \geq 0$  (positive-definite)

• Def. We say that  $f, g \in \mathcal{R}$  are orthogonal if

$$\langle f, g \rangle = 0$$

In this case, we write  $f \perp g$ .

• Def (norm) For  $f \in \mathcal{R}$ , set

$$\|f\| = \sqrt{\langle f, f \rangle}$$

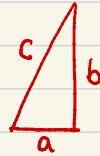
$$= \sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx}$$

We call  $\|f\|$  the norm of  $f$  (or the  $L^2$ -norm of  $f$ ).

Lem 1.2 ① (Pythagorean Thm)

If  $f \perp g$ , then

$$\|f+g\|^2 = \|f\|^2 + \|g\|^2.$$



② (Cauchy-Schwartz inequality)

$$|\langle f, g \rangle| \leq \|f\| \cdot \|g\|.$$

③ (triangle inequality)

$$\|f+g\| \leq \|f\| + \|g\|.$$

Pf. Refer to any textbook on linear algebra.  $\square$

For  $n \in \mathbb{Z}$ , define

$$e_n(x) := e^{inx}, \quad x \in [-\pi, \pi]$$

Then  $e_n \in \mathcal{R}$ .

Observe that for  $f \in \mathcal{R}$ ,

$$\begin{aligned}\hat{f}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \\ &= \langle f, e_n \rangle\end{aligned}$$

Notice that  $\{e_n\}_{n=-\infty}^{\infty} \subset \mathcal{R}$ , it is orthonormal in the following sense:

①  $\|e_n\| = 1$  for all  $n \in \mathbb{Z}$

②  $e_n \perp e_m$  if  $n \neq m$ .

Lem 1.3. For any  $N \in \mathbb{N}$  and any  $-N \leq n \leq N$ ,

$$\langle f - S_N f, e_n \rangle = 0$$

for all  $f \in \mathcal{R}$ .

Pf. Notice that

$$\begin{aligned} \langle f - S_N f, e_n \rangle &= \langle f, e_n \rangle - \langle S_N f, e_n \rangle \\ &= \widehat{f}(n) - \widehat{(S_N f)}(n) \\ &= \widehat{f}(n) - \widehat{f}(n) \\ &= 0 \end{aligned}$$

□

Corollary 1.4. Let  $f \in \mathcal{R}$ ,  $N \in \mathbb{N}$ , and let

$$p(x) = \sum_{n=-N}^N c_n e^{inx}$$

Then  $p \perp (f - S_N f)$ .

Lem 1.5. Let  $p(x) = \sum_{n=-N}^N c_n e^{inx} = \sum_{n=-N}^N c_n e_n$

Then

$$\|p\|^2 = \sum_{n=-N}^N |c_n|^2$$

Consequently

$$\|p\| = 0 \Leftrightarrow c_n = 0 \text{ for } -N \leq n \leq N$$

Pf.

$$\|p\|^2 = \langle p, p \rangle$$

$$= \left\langle \sum_{n=-N}^N c_n e_n, \sum_{m=-N}^N c_m e_m \right\rangle$$

$$= \sum_{n=-N}^N \sum_{m=-N}^N c_n \overline{c_m} \langle e_n, e_m \rangle$$

$$= \sum_{n=-N}^N |c_n|^2.$$



Lem 1.6 (Best approximation Thm)

Let  $f \in \mathcal{R}$  and  $N \in \mathbb{N}$ . Then

$$(1) \quad \|f - S_N f\| \leq \|f - \sum_{n=-N}^N c_n e_n\|$$

for any  $(c_n)_{-N}^N \in \mathbb{C}$ .

(2) The "=" holds if and only if  $c_n = \hat{f}(n)$   
for all  $-N \leq n \leq N$ .

$$\begin{aligned} \text{pf.} \quad f - \sum_{n=-N}^N c_n e_n &= (f - S_N f) + (S_N f - \sum_{n=-N}^N c_n e_n) \\ &= (f - S_N f) + \underbrace{\sum_{n=-N}^N (\hat{f}(n) - c_n) e_n}_{=: p} \end{aligned}$$

Notice that  $p \perp (f - S_N f)$  (by Cor 1.4).

By Pythagorean Thm,

$$\|f - \sum_{n=-N}^N c_n e_n\|^2 = \|f - S_N f\|^2 + \|p\|^2$$



$$= \|f - S_N f\|^2 + \sum_{-N}^N |\hat{f}^{(n)} - c_n|^2.$$

Hence  $\|f - \sum_{-N}^N c_n e_n\| \geq \|f - S_N f\|$

and " $\Leftarrow$ " holds  $\Leftrightarrow \hat{f}^{(n)} = c_n$  for  $-N \leq n \leq N$ .

□

Thm 1.1. Let  $f \in \mathcal{R}$ . Then

①  $\int_{-\pi}^{\pi} |f(x) - S_N f(x)|^2 dx \rightarrow 0$  as  $N \rightarrow \infty$ .

②  $\|f\|^2 = \sum_{-\infty}^{\infty} |\hat{f}^{(n)}|^2$ .

Pf. We first prove ①, which is equivalent to

$$\|f - S_N f\| \rightarrow 0 \text{ as } N \rightarrow \infty.$$

We claim that for any  $\varepsilon > 0$ ,  $\exists$  a trigonometric polynomial  $p$  such that

$$\|f - p\| < \varepsilon.$$

We first assume that  $f$  is cts on the circle.

In this case, by the Weierstrass approximation Thm,

$\exists$  a trigonometric poly  $p$  such that

$$|f(x) - p(x)| < \varepsilon \quad \text{for all } x \in [-\pi, \pi].$$

Then

$$\begin{aligned} \|f - p\|^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - p(x)|^2 dx \\ &\leq \varepsilon^2 \end{aligned}$$

Hence

$$\|f - p\| < \varepsilon.$$

Next we consider the general case when  $f \in \mathcal{R}$ .

In such case, we can find a cts function  $g$  on the circle such that

$$\bullet \quad \sup_{x \in [-\pi, \pi]} |g(x)| \leq \sup_{x \in [-\pi, \pi]} |f(x)| =: \|f\|_{\infty}$$

$$\bullet \quad \int_{-\pi}^{\pi} |f(x) - g(x)| dx \leq \frac{\varepsilon^2}{4 \cdot \|f\|_{\infty}}$$

$$\begin{aligned}
\text{Then } \|f - g\|^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - g(x)|^2 dx \\
&\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} 2\|f\|_{\infty} \cdot |f(x) - g(x)| dx \\
&\leq \frac{\|f\|_{\infty}}{\pi} \int_{-\pi}^{\pi} |f(x) - g(x)| dx \\
&\leq \frac{\|f\|_{\infty}}{\pi} \cdot \frac{\varepsilon^2}{4 \cdot \|f\|_{\infty}} \\
&= \frac{\varepsilon^2}{4\pi} < \frac{\varepsilon^2}{4}
\end{aligned}$$

$$\text{Hence } \|f - g\| < \frac{\varepsilon}{2}.$$

Then we can find a trigonometric poly  $P$  such that

$$\|g - P\| < \frac{\varepsilon}{2}$$

By the triangle inequality

$$\begin{aligned}
\|f - P\| &\leq \|f - g\| + \|g - P\| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This completes the proof of the claim.

Let  $P$  be given as in the claim.

Write  $M = \deg(P)$ .

Next we show that

$$\|f - S_N f\| < \varepsilon \quad \text{for all } N \geq M. \quad (*)$$

Since  $N \geq M$ , by the Best approximation Thm

$$\|f - S_N f\| \leq \|f - P\| < \varepsilon.$$

That is,  $(*)$  holds. Therefore

$$\lim_{N \rightarrow \infty} \|f - S_N f\| = 0.$$

Now we prove ② :  $\|f\|^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$ .

Notice that  $f = (f - S_N f) + S_N f$

But  $S_N f \perp (f - S_N f)$ . (by Cor 1.4).

By Pythagorean Thm,

$$\begin{aligned}\|f\|^2 &= \|f - S_N f\|^2 + \|S_N f\|^2 \\ &= \|f - S_N f\|^2 + \sum_{n=-N}^N |\hat{f}(n)|^2\end{aligned}$$

Letting  $N \rightarrow \infty$  and observing that  
 $\|f - S_N f\| \rightarrow 0$ ,

hence

$$\begin{aligned}\|f\|^2 &= \lim_{N \rightarrow \infty} \sum_{n=-N}^N |\hat{f}(n)|^2 \\ &= \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2.\end{aligned}$$



Remark: In Thm 1.1,  $f \in \mathcal{R}$  can be replaced by any integrable function on  $[-\pi, \pi]$ .

Example 1.7. Let  $f(x) = x$  on  $[-\pi, \pi]$ .

$$f(x) \sim \sum_{n \neq 0} \frac{(-1)^n i}{n} e^{inx}$$

By the Parseval identity, we have

$$\begin{aligned} \|f\|^2 &= \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \\ &= \sum_{n \neq 0} \frac{1}{n^2} = 2 \cdot \sum_{n=1}^{\infty} \frac{1}{n^2} \end{aligned}$$

But

$$\begin{aligned} \|f\|^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx \\ &= \frac{1}{2\pi} \cdot \left. \frac{x^3}{3} \right|_{-\pi}^{\pi} = \frac{\pi^2}{3}. \end{aligned}$$

So we have

$$\frac{\pi^2}{3} = 2 \cdot \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \cdot (\text{Euler's identity}),$$